

MATH 2050 C Lecture 12 (Feb 23)

[Problem Set 6 posted, due on Mar 3.]

Recall: $(x_n)_{n \in \mathbb{N}} \rightsquigarrow$ subseq. $(x_{n_k})_{k \in \mathbb{N}}$

Thm: Suppose $\lim_{n \rightarrow \infty} x_n = x$. Then, every subseq. (x_{n_k}) of (x_n) also converges to the same limit, i.e. $\lim_{k \rightarrow \infty} x_{n_k} = x$.

Proof: Note: $n_k \geq k$ for all $k \in \mathbb{N}$ (by induction).

Let $\varepsilon > 0$ be fixed but arbitrary.

$$\lim_{n \rightarrow \infty} x_n = x \Rightarrow \exists K \in \mathbb{N} \text{ s.t. } |x_n - x| < \varepsilon \quad \forall n \geq K$$

By Note above, when $k \geq K$, then $n_k \geq k \geq K$. Thus,

$$|x_{n_k} - x| < \varepsilon \quad \forall k \geq K$$

Example: Show that $\lim_{n \rightarrow \infty} C^{\frac{1}{n}} = 1$ for $C > 1$.

Pf: Let $z_n := C^{\frac{1}{n}}$. Then, by induction,

(z_n) is decreasing and bdd below by 1

By MCT, $\lim_{n \rightarrow \infty} (z_n) =: z$ exists.

Consider the subseq. $(z_{n_k})_{k \in \mathbb{N}} = (z_{2k})$, by Thm above,

$$\lim_{k \rightarrow \infty} (z_{n_k}) = z.$$

$$\text{Now, } z_{2n} = C^{\frac{1}{2n}} = (C^{\frac{1}{n}})^{\frac{1}{2}} = (z_n)^{\frac{1}{2}}$$

$\therefore z_n > 1 \quad \forall n \in \mathbb{N}$
rejected

Take $n \rightarrow \infty$ on both sides, we have $z = \sqrt{z} \Rightarrow z = \emptyset$ or 1.

In summary.

MCT: (x_n) monotone + bdd $\Rightarrow (x_n)$ convergent.

Thm: (x_n) convergent $\Rightarrow (x_n)$ bdd

Thm: (x_n) convergent \Rightarrow ANY subseq. (x_{n_k}) of (x_n)
converge to the SAME limit.

Take negation yields two divergence criteria:

Cor: (x_n) unbdd $\Rightarrow (x_n)$ divergent

Cor: Either: \exists subseq (x_{n_k}) which is divergent
or: \exists two subseq (x_{n_k}) and (x_{n_i}) s.t

$$\lim_{k \rightarrow \infty} (x_{n_k}) \neq \lim_{i \rightarrow \infty} (x_{n_i})$$

$\Rightarrow (x_n)$ divergent.

Example: $(-1)^n$ is divergent since \exists two subseq.

$$(1, 1, 1, 1, \dots, 1) \rightarrow 1$$

$$(-1, -1, -1, -1, \dots, -1) \rightarrow -1$$

Example: $(\cos \frac{n\pi}{2}) = (0, -1, 0, 1, 0, -1, 0, 1, \dots)$

\exists subseq. $(0, 0, \dots, 0) \rightarrow 0$.

$(-1, 1, -1, 1, \dots)$ divergent \Rightarrow original seq is divergent.

Example: $(x_n) = (0, 1, 0, 2, 0, 3, 0, \dots, 0, n, \dots)$ divergent

since \exists subseq. $(1, 2, 3, 4, \dots, n, \dots)$ unbdd \Rightarrow divergent.

Recall: (x_n) divergent $\Leftrightarrow (x_n)$ DOES NOT converge to x for ANY $x \in \mathbb{R}$.

Thm: Fix $x \in \mathbb{R}$. Then

(x_n) does NOT converge to x either (x_n) divergent
or $(x_n) \rightarrow x' \neq x$

$\Leftrightarrow \exists \varepsilon_0 > 0$ AND a subseq. (x_{n_k}) of (x_n) s.t.

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \forall k \in \mathbb{N}.$$

Proof: Recall:

$$\lim_{n \rightarrow \infty} (x_n) = x \quad \Leftrightarrow \quad \forall \varepsilon > 0, \exists K = K(\varepsilon) \in \mathbb{N} \text{ s.t.} \\ |x_n - x| < \varepsilon \quad \forall n \geq K$$

Negate the above.

$$(x_n) \text{ does NOT converge to } x \quad \Leftrightarrow \quad \exists \varepsilon_0 > 0 \text{ s.t. } \forall K \in \mathbb{N} \text{ s.t.} \\ \exists n_k \geq K \text{ s.t. } |x_{n_k} - x| \geq \varepsilon_0$$

• Take $K = 1$, choose $n_1 \geq 1$ s.t. $|x_{n_1} - x| \geq \varepsilon_0$

• Take $K = n_1 + 1$, choose $n_2 \geq n_1 + 1$ s.t. $|x_{n_2} - x| \geq \varepsilon_0$

repeat $\leadsto (x_{n_k})_{k \in \mathbb{N}}$ s.t. $|x_{n_k} - x| \geq \varepsilon_0$

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